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Abstract

This paper deals with polynomial cointegration, i.e. with the phenomenon that linear combinations of a vector valued rational unit root process and lags of the process are of lower integration order than the process itself (for definitions see Section 2). The analysis is performed in the state space representation of rational unit root processes derived in Bauer and Wagner (2003). The state space framework is an equivalent alternative to the ARMA framework. Unit roots are allowed to occur at any point on the unit circle with arbitrary integer integration order. In the paper simple criteria for the existence of non-trivial polynomial cointegrating relationships are given. Trivial cointegrating relationships lead to the reduction of the integration order simply by appropriate differencing. The set of all polynomial cointegrating relationships is determined from simple orthogonality conditions derived directly from the state space representation. These results are important for analyzing the structure of unit root processes and their polynomial cointegrating relationships and also for the parameterization for system sets with given cointegration properties.

JEL Classification: C13, C32
Keywords: Unit roots, polynomial cointegration, state space representation

1 Introduction

Polynomial cointegration, first introduced as multi-cointegration in Yoo (1986) and Granger and Lee (1989a; 1989b) is a natural generalization of cointegration. Cointegration describes the fact that for multivariate integrated processes (i.e. processes that can be transformed to stationarity by appropriate differencing, see Definition 1 below) there may exist linear combinations $\beta' y_t$, which are integrated of lower order than y_t itself or even stationary. In polynomial cointegration analysis not only static (linear) combinations of the variables are studied, but the relationships are extended to include lagged variables, resulting in transformed processes of the form $\sum_{j=0}^{q} \beta_j' y_{t-j}$, compactly written as $\beta(z)' y_t$, where $\beta(z) = \sum_{j=0}^{q} \beta_j z^j$ and z denotes the backward shift operator as well as a complex variable (see Section 2 for precise definitions).

The analysis in this paper is based on the state space framework, which is in a sense made precise in Bauer and Wagner (2003), equivalent to the ARMA framework for the representation of unit root processes. State space systems can be used to obtain very convenient representations of stochastic processes with unit roots with integer integration orders at finitely many arbitrary points on the unit circle. The representation result is based upon the canonical state space representation developed in Bauer and Wagner (2003). In that paper a specific canonical form is developed that clearly reveals the integration and cointegration properties of the underlying process. The present paper is concerned with showing that the developed canonical state space representation also directly leads to a simple and convenient representation of all polynomial cointegrating relationships via orthogonality constraints. The main ingredient is an appropriate more thorough investigation of the previously developed representation results.

The results derived below are comparable to the representation results derived in Gregoir (1999), based on the Wold representation of the sufficiently differenced process. Gregoir's results are, as most contributions in the (co)integration literature, formulated in the ARMA framework. Some other contributions on the representation of integrated processes in the ARMA framework, however restricted to unit roots only at $z = 1$, are e.g.: Gregoir and Laroque (1994), which is also based on the Wold representation and can be seen as a predecessor of Gregoir (1999); Stock and Watson (1993), who base their analysis of higher order integrated processes on a triangular representation; Haldrup and Salmon (1998), who base their representation of I(2) processes on the Smith-McMillan form and Deistler and Wagner (2000) who base their investigation of integrated systems on the transfer function. Comparing our results with the existing ARMA based representation results for polynomial cointegration, we are led to conclude that the state space framework is suited better for obtaining an understanding of the structure of polynomially cointegrated systems. This understanding is important in two respects: The results in this paper can be used to reveal the cointegration properties of a given system. Secondly, and probably more importantly, the results in this paper can be used to derive parameterizations for systems with given polynomial cointegration properties incorporated. To the best of our knowledge, this is not possible in the ARMA framework with the results available in the literature.

The paper is organized as follows: In section 2 the model set, the assumptions and some definitions are presented. Section 3 discusses the state space framework. Section 4 discusses the links between complex and real valued system representations and in section 5 the basic ideas are illustrated with a small I(2) example. In section 6 polynomial cointegrating relationships are discussed in the state space representation and in section 7 the results of the preceding section are sharpened to focus on the *relevant*, i.e. non-trivial and minimum degree, polynomial cointegrating relationships. In section 8 an example to illustrate the results of section 7 is discussed and section 9 briefly summarizes and concludes the paper. The appendix provides additionally a convenient and intuitive representation and interpretation of polynomial cointegration when focusing only on one unit root.

Throughout the paper we denote with I_n the $n \times n$ identity matrix and with $0^{a \times b}$ the null-matrix of dimensions $a \times b$. Conjugate complex numbers are denoted by $\bar{x}$ and X' denotes the Hermite transpose of a matrix X. Throughout $\lambda_{max}(A)$ denotes an eigenvalue of maximum modulus of the matrix A.

2 Definitions and Assumptions

This section is devoted to present the required definitions and assumptions concerning unit root processes, their unit root structure and (polynomial) cointegration. This in turn requires in a first step to define the differencing operator at frequency ω and *linearly deterministic* processes.

The difference operator at frequency ω is defined as follows:

$$\Delta_\omega(z) = \begin{cases} 1 - e^{i\omega}z, & \omega \in \{0, \pi\} \\ (1 - e^{i\omega}z)(1 - e^{-i\omega}z), & \omega \in (0, \pi). \end{cases} \tag{1}$$

Here z denotes a complex function as well as the backward shift operator. Further let $\Delta := \Delta_0(z)$ to simplify notation. The way we define the differencing operator $\Delta_\omega(z) = (1-e^{i\omega}z)(1-e^{-i\omega}z) = 1-2(\cos\omega)z+z^2$ for $\omega \in (0,\pi)$ incorporates the assumption of real valued y_t by filtering pairs of complex conjugate roots: For real valued processes complex roots are occurring in pairs of complex conjugate roots. Note that in order to apply the differencing operator to a process defined on $\mathbb{N}$, initial conditions have to be specified.

A process $(d_t; t \in \mathbb{N})$ is called *linearly deterministic*, if it is perfectly predictable from its own past from some time instant t_0 onwards: Let $d_{t|t_0}$ denote the best linear least squares prediction of d_t based on $d_j, j = 1, \dots, t_0$. Then d_t is said to be linearly deterministic, if there exists a $t_0 \in \mathbb{N}$, such that $\sup_{t>t_0} \mathbb{E}\|d_{t|t_0} - d_t\| = 0$. Thus, for instance any solution to a vector difference equation $\sum_{j=0}^{p} A_j d_{t-j} = 0, t \in \mathbb{N}$ for some matrices $A_j \in \mathbb{R}^{s\times s}, j = 0, \dots, p$ is a linearly deterministic process. Therefore, not surprising, e.g. constants, linear or polynomial trends

and seasonal dummies are linearly deterministic processes.

We are now ready to define the unit root structure.

Definition 1 *The s-dimensional real random process $(y_t; t \in \mathbb{N})$ has unit root structure $((\omega_1, h_1), \dots, (\omega_{l^\mathbb{R}}, h_{l^\mathbb{R}}))$ with $0 \leq \omega_1 < \omega_2 < \dots < \omega_{l^\mathbb{R}} \leq \pi, h_k \in \mathbb{N}, k = 1, \dots, l^\mathbb{R}$, if there exist random initial values $y_{1-H}, \dots, y_0, H = \sum_{k=0}^{l^\mathbb{R}} (h_k + h_k \mathbb{I}(\omega_k \notin \{0, \pi\}))$ with finite second moments and a linearly deterministic process $(T_t; t \in \mathbb{N})$ such that*

$$\Delta_{\omega_1}^{h_1}(z) \dots \Delta_{\omega_{l^\mathbb{R}}}^{h_{l^\mathbb{R}}}(z) y_t = v_t + T_t, \quad t \in \mathbb{N} \tag{2}$$

for $v_t = \sum_{j=0}^{\infty} c_j \varepsilon_{t-j}$ corresponding to the Wold representation of the stationary process $(v_t; t \in \mathbb{Z}), v_t \in \mathbb{R}^s$, where for $c(z) = \sum_{j=0}^{\infty} c_j z^j$ with $\sum_{j=0}^{\infty} \|c_j\| < \infty$ it holds that $c(e^{i\omega_k}) \neq 0, k = 1, \dots, l^\mathbb{R}$. Here $\mathbb{I}(.)$ denotes the indicator function.

The s-dimensional random process $(y_t; t \in \mathbb{N})$ has complex unit root structure $((\omega_1, h_1), \dots, (\omega_l, h_l))$ with $z_k = e^{i\omega_k}$, $0 \leq \omega_1 < \omega_2 < \dots < \omega_l < 2\pi$ and $h_k \in \mathbb{N}$ for $k = 1, \dots, l$, if there exist random initial conditions $y_{1-H}, \dots, y_0, H = h_1 + \dots + h_l$ with finite second moments and *a linearly deterministic process* $(T_t; t \in \mathbb{N})$ *such that*

$$\prod_{k=1}^{l}(1 - z_k z)^{h_k} y_t = v_t + T_t, \quad t \in \mathbb{N} \tag{3}$$

with $v_t = c(z)\varepsilon_t \in \mathbb{C}^s$ *corresponding to the Wold decomposition of the stationary process* $(v_t; t \in \mathbb{Z})$ *for* $c(z) = \sum_{j=0}^{\infty} c_j z^j$ *with* $\sum_{j=0}^{\infty} \|c_j\| < \infty$ *and it holds that* $c(e^{i\omega_k}) \neq 0, k = 1, \ldots, l$.

If $c(z)$ *is a rational function of* z*, then* y_t *is called* rational process.

The unit root structure is defined for the multivariate process and not componentwise. Consequently not each component of the process needs to have the same unit root structure. This is implied by requiring $c(z_k) \neq 0$ rather than $\det c(z_k) \neq 0$. The order of integration h_k at the unit root z_k denotes the maximum order of integration of the components of $(y_t; t \in \mathbb{N})$ at z_k. Note further that the definition excludes fractionally integrated processes: The summability condition prevents v_t to be fractionally integrated of order $d \in (0, .5)$. Processes with fractional integration order $d \in [0.5, 1)$ at some unit root $z_k = e^{i\omega_k}$ are nonstationary and therefore have to be differenced once to transform the process to stationarity, which implies that the corresponding function $c(z)$ can be factorized as $c(z) = \Delta_{\omega_k}(z)^{1-d}\tilde{c}(z)$ which implies $c(z_k) = 0$.

Note furthermore that the restriction $c(z_k) \neq 0$ only at the unit roots allows to classify processes, which have been overdifferenced at some points on the unit circle while still containing unit roots at other locations. Finally note that the inclusion of deterministic terms in the definition implies that e.g. so called trend stationary processes are not integrated. Also the first difference of a process with unit root structure $((0, 1))$ need not be stationary, but only stationary up to linearly deterministic processes. In later sections, to distinguish notationally, we will use the term *complex integrated of order* h_k *at* $z_k = e^{i\omega_k}$, if the process y_t has a complex unit root structure that includes the pair (ω_k, h_k).

In this paper we restrict attention to cointegration with real valued cointegrating relationships and base the definition of cointegration on the unit root structure. This implies, see also the discussion below, that we consider in the case of pairs of complex conjugate unit roots, cointegrating relationships that reduce the *complex* integration order corresponding to both members of the pair of unit roots by an equal number. The connection to complex cointegration, which allows to consider each unit root separately, will be only remarked upon as the developed results directly allow to consider this case as well.

For processes with higher integration orders and with unit roots at a variety of points on the unit circle a multitude of possibilities for cointegration and polynomial cointegration of different orders arises. For the vector polynomial $\beta(z) = \sum_{j=0}^{q} \beta_j z^j, \beta_j \in \mathbb{R}^s$ let $\beta(z)' y_t = \sum_{j=0}^{q} \beta_j' y_{t-j}$, where $y_t = 0$ for $t < 1$ is used. In the following definition we use the understanding that pairs $(\omega_k, 0)$ are removed from the unit root structure of the transformed process.

Definition 2 *A random process $(y_t; t \in \mathbb{N})$ with unit root structure $((\omega_1, h_1), \ldots, (\omega_{l^{\mathbb{R}}}, h_{l^{\mathbb{R}}}))$ is called* cointegrated or statically cointegrated of order $\big((\omega_1, h_1, h_1^p), \ldots, (\omega_{l^{\mathbb{R}}}, h_{l^{\mathbb{R}}}, h_{l^{\mathbb{R}}}^p)\big)$, $0 \leq h_k^p \leq h_k, k = 1, \ldots, l^{\mathbb{R}}$, $\max_{k=1,\ldots,l^{\mathbb{R}}}(h_k - h_k^p) > 0$, *if there exists a vector $\beta \in \mathbb{R}^s, \beta \neq 0$ such that $(\beta' y_t; t \in \mathbb{N})$ has unit root structure $\big((\omega_1, h_1^p), \ldots, (\omega_{l^{\mathbb{R}}}, h_{l^{\mathbb{R}}}^p)\big)$. The vector β is in this case called cointegrating vector of order $\big((\omega_1, h_1, h_1^p), \ldots, (\omega_{l^{\mathbb{R}}}, h_{l^{\mathbb{R}}}, h_{l^{\mathbb{R}}}^p)\big)$.*

A random process $(y_t; t \in \mathbb{N})$ with unit root structure $((\omega_1, h_1), \ldots, (\omega_{l^{\mathbb{R}}}, h_{l^{\mathbb{R}}}))$ is called polynomially cointegrated of order $\big((\omega_1, h_1, h_1^p), \ldots, (\omega_{l^{\mathbb{R}}}, h_{l^{\mathbb{R}}}, h_{l^{\mathbb{R}}}^p)\big)$, $0 \leq h_k^p \leq h_k, k = 1, \ldots, l^{\mathbb{R}}$, with $\max_{k=1,\ldots,l^{\mathbb{R}}}(h_k - h_k^p) > 0$, *if there exists a vector polynomial $\beta(z) = \sum_{j=0}^{q} \beta_j z^j, \beta_j \in \mathbb{R}^s$, with* $\max_{k=1,\ldots,l^{\mathbb{R}}} \|\beta(z_k)\|(h_k - h_k^p) > 0$ *and $\beta(0) \neq 0$, such that $(\beta(z)' y_t; t \in \mathbb{N})$ has unit root structure $\big((\omega_1, h_1^p), \ldots, (\omega_{l^{\mathbb{R}}}, h_{l^{\mathbb{R}}}^p)\big)$, with $0 \leq h_k^p \leq h_k, k = 1, \ldots, l^{\mathbb{R}}$. The vector polynomial $\beta(z)$ is in this case called polynomial cointegrating vector of order $\big((\omega_1, h_1, h_1^p), \ldots, (\omega_{l^{\mathbb{R}}}, h_{l^{\mathbb{R}}}, h_{l^{\mathbb{R}}}^p)\big)$.*

Remark 1 *As mentioned above, cointegration and polynomial cointegration can easily be extended to complex cointegration and complex polynomial cointegration, using the complex unit root structure as the basis in Definition 2 and allowing for complex coefficients $\beta_j \in \mathbb{C}^s$. Complex cointegration allows to investigate cointegration with respect to each unit root separately, results however in general in complex valued transformed processes. In section 4 we will briefly discuss some of the differences that occur between a real and a complex valued perspective on cointegration.*

Remark 2 *Note that in the definition of the unit root structure the existence of appropriate initial conditions has been postulated, whereas in the definition of the polynomial filter $\beta(z)$ we assume zero initial conditions irrespective of the true initial conditions.*

The specific choice of initial conditions in defining the polynomial filters is however not *critical. This stems from the fact that filtered processes $\beta(z)'y_t$ for different choices of the initial conditions differ only in the first q time instants. This difference can be included in the linearly deterministic process $(T_t; t \in \mathbb{N})$, since it is obvious that a process that is non-zero only for $t = 1, \ldots, q$ is linearly deterministic.*

Remark 3 *Our definition of polynomial cointegration differs from the Definition 3.1. in Gregoir (1999) by considering the change in the unit root structure of the transformed process rather than only the difference in the integration order at one unit root. A second difference is the exclusion of trivial cointegrating polynomials, which reduce the integration order only by differencing, see the definition below. Thirdly, Gregoir (1999) defines the order of cointegration based on the polynomial degree of the cointegrating polynomial, whereas our definition is based on the reduction of the integration orders.*

Definition 3 (Triviality) *A polynomial vector $\beta(z) = \sum_{j=0}^{q} \beta_j z^j$, $\beta_j \in \mathbb{R}^s$ is called trivial, if $\max_{k=1,\ldots,l^{\mathbb{R}}} \|\beta(z_k)\|(h_k - h_k^p) = 0$ or if $\beta(0) = 0$ holds. Note that trivial polynomial cointegration vectors have already been excluded in Definition 3.*

Hence, *non-trivial* polynomial cointegrating vectors reduce the integration order for at least one unit root not just due to differencing at that unit root. A remark in this respect is that only the maximum of $\|\beta(z_k)\|(h_k - h_k^p)$ over all unit roots has to be positive. This implies that for all but one unit root the reduction in the integration order is allowed to be achieved by applying suitable multiples of the respective differencing filters.

A second source of redundancy in the set of polynomial cointegrating relationships is the polynomial degree of the polynomial cointegrating vector. Given a non-trivial polynomial cointegrating vector it is easily possible to increase the polynomial degree without changing the order of the cointegrating relationship. Consider as simplest examples the multiplication of a non-trivial polynomial cointegrating relationship $\beta(z)$ of order $\big((\omega_1, h_1, h_1^p), \ldots, (\omega_{l^{\mathbb{R}}}, h_{l^{\mathbb{R}}}, h_{l^{\mathbb{R}}}^p)\big)$ with scalar polynomials, say $p(z)$, to arrive at e.g. $p(z)\beta(z)$ or by adding any polynomial of the form $\gamma\Delta_{\omega_1}(z)^{h_1-h_1^p} \ldots \Delta_{\omega_l^{\mathbb{R}}}(z)^{h_{l^{\mathbb{R}}}-h_{l^{\mathbb{R}}}^p}$, $\gamma \neq 0$ to $\beta(z)$. To exclude such cases of polynomial cointegrating relationships that do not add additional insights compared to polynomial cointegrating relationships of lower polynomial degree, the following definition of *minimum-degree* polynomial cointegrating relationship is used. Introduce a semi-ordering of unit root structures as follows: Let $\Theta = ((\omega_1, h_1), \ldots, (\omega_{l^{\mathbb{R}}}, h_{l^{\mathbb{R}}}))$

and $\tilde{\Theta} = ((\omega_1, \tilde{h}_1), \ldots, (\omega_{l^{\mathbb{R}}}, \tilde{h}_{l^{\mathbb{R}}}))$ be two unit root structures, where possibly zero entries $h_k, \tilde{h}_k$ are allowed for in order to compare all unit roots occurring in both structures $\Theta, \tilde{\Theta}$. Then $\Theta \preceq \tilde{\Theta} \Leftrightarrow \max_{k=1,\ldots,l^{\mathbb{R}}}(h_k - \tilde{h}_k) \leq 0$, i.e. $h_k \leq \tilde{h}_k$ for each unit root.

Definition 4 (Non-Minimum Degree) *A non-trivial polynomial cointegrating vector $\beta(z) = \sum_{j=0}^{q} \beta_j z^j$ of order $((\omega_1, h_1, h_1^p), \ldots, (\omega_l, h_l, h_l^p))$ is said to be of non-minimum degree, if there exists a representation $\beta(z) = \sum_{j=1}^{m} p_j(z)\beta_j(z)$ for some finite integer m, where*

1. *$p_j(z)$ are scalar polynomials,*

2. *$\beta_j(z)$ are vector polynomials, such that the degrees of the scalar polynomials in its entries are smaller or equal than the degrees of the entries in $\beta(z)$ with strict inequality for at least one entry,*

3. *the polynomial degree of $p_j(z)\beta_j(z)$ is smaller or equal to the polynomial degree of $\beta(z)$,*

4. *$(p_j(z)\beta_j(z)'y_t; t \in \mathbb{N})$ has unit root structure $\tilde{\Theta} \preceq ((\omega_1, h_1^p), \ldots, (\omega_l, h_{l^{\mathbb{R}}}^p))$.*

It is easy to check that the examples given before the definition are all non-minimum degree polynomials. Non-minimum degree polynomials are seen to be redundant, as they do not add to the understanding of the cointegration properties of the process. Therefore it follows that minimum degree polynomials are of maximum degree $\sum_{k=1}^{l^{\mathbb{R}}}(h_k - h_k^p)(1 + \mathbb{I}(z_k \neq \pm 1)) - 1$.

3 State Space Framework

As in the companion paper Bauer and Wagner (2003), also in this paper we consider rational processes in their state space representation. I.e. we consider processes that can be represented as the solution to the state space system equations:

$$\begin{aligned} y_t &= Cx_t + d_t + \varepsilon_t, \\ x_{t+1} &= Ax_t + B\varepsilon_t, \end{aligned} \tag{4}$$

where $(y_t; t \in \mathbb{N})$ denotes the s-dimensional output process. $(\varepsilon_t; t \in \mathbb{Z})$ denotes an s-dimensional unobserved white noise sequence, which is here for simplicity assumed to be i.i.d. $x_t \in \mathbb{C}^n$ denotes the n-dimensional unobserved state vector and $d_t \in \mathbb{R}^s$ is a linearly deterministic process. $A \in \mathbb{C}^{n\times n}, B \in \mathbb{C}^{n\times s}, C \in \mathbb{C}^{s\times n}$ are complex matrices, however the corresponding impulse response coefficients $K_j = CA^{j-1}B$ for $j > 0, K_0 = I_s$ are restricted to be real valued, since we are only interested in real valued output processes $(y_t; t \in \mathbb{N})$. The computations are performed using complex quantities to simplify the algebra and the required notation. If one prefers, all computations can equivalently be performed using real valued quantities, see Bauer and Wagner (2003) or also the following section for details on the relation between complex and real valued representations. The recursions are started at the initial state $x_1 \in \mathbb{C}^n$, which is restricted in order to obtain a real valued output. x_1 is assumed to be a random variable with finite variance uncorrelated with the noise $(\varepsilon_t; t \in \mathbb{N})$. Note that this also includes the case of a constant initial state.

In the following we provide a very brief presentation of some of the main properties of state space systems relevant for the paper. The intention is merely to provide a list of keywords for the state space analogues of concepts well known in the ARMA framework. Readers interested in more detailed discussions on state space systems are referred to Hannan and Deistler (1988), in particular Chapters 1 and 2.

The state space system (A, B, C) (cf. equations (4)) corresponds to a transfer function $k(z) = I_s + zC(I_n - zA)^{-1}B =: \Pi(A, B, C)$, where z here denotes a scalar complex variable. This equation defines the mapping Π. Note that by construction $k(z)$ is a rational function. Conversely, also for each rational function $k(z)$ with $k(0) = I_s$, there exist state space realizations, i.e. there exist matrix triples (with appropriate dimensions of the matrices) (A, B, C) such that $k(z) = \Pi(A, B, C)$, see Hannan and Deistler (1988, Chapter 1). The matrix triple (A, B, C) is often referred to as *state space realization* of $k(z)$.

A similar link prevails also between ARMA systems and rational transfer functions. For all rational functions $k(z)$ with $k(0) = I_s$, there exist matrix fraction decompositions with left coprime matrix polynomials $a(z) = \sum_{j=0}^{p} A_j z^j, A_0 = I_s, A_p \neq 0$, $b(z) = \sum_{j=0}^{q} B_j z^j, B_0 = I_s, B_q \neq 0$, such that $k(z) = a^{-1}(z)b(z)$ holds. Analogous to the above, the pair $(a(z), b(z))$ is referred to as an *ARMA realization* of the transfer function. Combining the two links it immediately follows that for every ARMA system $(a(z), b(z))$ there exist state space systems (A, B, C) such that $\Pi(A, B, C) = a^{-1}(z)b(z)$.

A state space realization (A, B, C) of a given transfer function $k(z)$ is called *minimal*, if there exists no other state space realization of $k(z)$ with a smaller state dimension. Minimality is the state space analogue to left coprimeness. From the ARMA framework it is well understood that in a left coprime realization the locations of the roots of the determinant of the matrix polynomial $a(z)$ determine the integration or stationarity properties of the resulting ARMA processes. The analogue for minimal state space realizations are the locations of the eigenvalues of A: If the poles of $k(z)$ are defined as the roots of $\det a(z)$ from any left coprime matrix fraction description $(a(z), b(z)), k(z) = a^{-1}(z)b(z)$, then λ is a pole of $k(z)$ if and only if $\det(I_n - \lambda A) = 0$ for any minimal state space realization (A, B, C) of $k(z)$ (cf. e.g. Hannan and Deistler, 1988, Theorem 1.2.2.). Hence, if $\lambda \neq 0$ is a pole of $k(z)$, then λ^{-1} is an eigenvalue of A. Similarly, if the zeros of the transfer function are defined as the zeros of $\det b(z)$, then λ is a zero of $k(z)$, if and only if $\det(I_n - \lambda(A - BC)) = 0$. The paper deals only with processes with eigenvalues of A smaller or equal than one in absolute value, this restriction of $|\lambda_{max}(A)| \leq 1$ is called *non-explosiveness* restriction. In terms of an ARMA representation we thus assume $\det(a(z)) \neq 0, |z| < 1$. Similarly we restrict attention to minimum-phase systems, i.e. to systems where $|\lambda_{max}(A - BC)| \leq 1$, or equivalently to transfer functions $k(z)$ such that the zeros of $k(z)$ lie outside the open unit disc.

Both, state space as well as ARMA realizations of a transfer function $k(z)$ are not unique. For a given minimal realization (A_0, B_0, C_0) of a given transfer function, the set of all minimal state space systems realizing the same transfer function is given by $\{(A, B, C) : \exists$ nonsingular $T \in \mathbb{C}^{n \times n} : A = TA_0T^{-1}, B = TB_0, C = C_0T^{-1}\}$. Hence, for any given transfer function there is some freedom to choose a minimal state space realization. This freedom can be exploited to select or construct a realization that highlights the properties most important in the context studied, in our case the integration and cointegration properties.

We have seen above that the state space and ARMA framework are equivalent in the sense that they are both capable of realizing the class of rational transfer functions. This is however not the only level at which equivalence can be established. It can also be shown that the solutions to the difference equation systems that constitute a state space or an ARMA system are closely related, see e.g. Hannan and Deistler (1988, Chapter 1). For the case of unit root processes defined on $\mathbb{N}$ this link has been investigated in Bauer and

Wagner (2003): A unit root process $(y_t; t \in \mathbb{N})$ is said to have a *state space representation*, if there exists a state space system (A, B, C), initial conditions x_1 and a linearly deterministic process $(d_t; t \in \mathbb{N})$ such that $(y_t; t \in \mathbb{N})$ is a solution of equations (4). Analogously we define an ARMA process to be the solution of

$$a(z)y_t = b(z)\varepsilon_t, t \in \mathbb{N} \tag{5}$$

for some polynomial matrices $(a(z), b(z))$ and (possibly random) initial conditions $y_j, j = 1-p, \dots, 0, \varepsilon_j, j = 1-q, \dots, 0$. Note that in the definition of an ARMA process no deterministic processes are explicitly taken into account. This implies that any linearly deterministic component, d_t say, present in the ARMA process y_t must be a solution of $a(z)d_t = 0$. It is shown in Hannan and Deistler (1988, p. 15), that for each ARMA process $(y_t; t \in \mathbb{N})$ there exist initial conditions x_1 and a (not necessarily minimal) state space system (A, B, C), such that $(y_t; t \in \mathbb{N})$ has a state space representation. Furthermore even $d_t = 0$ can be assumed without restriction of generality in (4).

Conversely, if a process has a minimal state space representation with $d_t = 0$, it can be shown that there exists a (not necessarily left coprime) ARMA system $(a(z), b(z))$, such that $(y_t; t \in \mathbb{N})$ satisfies the corresponding vector difference equation (5) for suitable initial conditions. Therefore in the absence of linearly deterministic processes $(d_t; t \in \mathbb{N})$, every solution to a minimal state space system can also be represented as a solution to an ARMA equation (5) and vice versa.

The above discussion requires minimality of the state space system. Representing the solution to the state space equations (4) as a function of the input and the initial state as

$$y_t = Cx_t + \varepsilon_t + d_t = \cdots = CA^{t-1}x_1 + d_t + \varepsilon_t + \sum_{j=1}^{t-1} CA^{j-1}B\varepsilon_{t-j},$$

shows that for the description of the impact of the noise on the output, minimality of the state space system can be assumed without restriction of generality (since the matrices CA^jB are invariant for all state space realizations of a transfer function). The additional term $CA^{t-1}x_1$ can be shown to be linearly deterministic and can therefore be attributed to d_t. Combining the above two arguments we can without restriction of generality state that minimal state space systems (A, B, C) are – for appropriate definition of $(d_t; t \in \mathbb{N})$ – capable of representing all ARMA processes $(y_t; t \in \mathbb{N})$ (cf. also Theorem 1).

The convenient representation of state space systems with a given complex unit root structure $((\omega_1, h_1), \ldots, (\omega_l, h_l))$ developed in Bauer and Wagner (2003) is also the main necessary ingredient for the representation of all polynomial cointegrating relationships. The developed canonical form starts from the already discussed observation that the eigenvalues of the A-matrix determine the integration properties of the solutions of the state space system. If all eigenvalues of A are smaller than one in absolute value, then there exist initial conditions x_1 such that the corresponding solution process is stationary (after removal of a possibly present linearly deterministic component d_t). Eigenvalues of modulus one correspond to unit roots. The structure of the eigenvalues, i.e. the sizes of the Jordan blocks corresponding to the Jordan segments (using the notation of Meyer, 2000) of A, determine the integration and cointegration properties of the solution process of the system. This statement is made precise in the following theorem which essentially summarizes the findings of Bauer and Wagner (2003). Note that for algebraic simplicity we consider a complex valued formulation. The transformation to obtain a real valued representation from this complex representation is discussed in the following section.

Theorem 1 *For each real rational process $(y_t; t \in \mathbb{N})$ with complex unit root structure $((\omega_1, h_1), \ldots, (\omega_l, h_l))$ a unique minimal state space representation (A, B, C) fulfilling the following restrictions exists:*

1. *The matrix A is block-diagonal: $A = diag(J_1, \ldots, J_l, A_{st})$. The diagonal blocks $J_k, k = 1, \ldots, l$ correspond to the Jordan segments of A corresponding to the unit roots $z_k = e^{i\omega_k}$, i.e. to the eigenvalues of modulus one, ordered according to increasing frequency ω_k. A_{st} accounts for the eigenvalues smaller than one in absolute value. Each Jordan segment J_k is in reordered Jordan normal form (see Bauer and Wagner, 2003):*

$$J_k = \begin{bmatrix} z_k I_{d_1^k} & [I_{d_1^k}, 0^{d_1^k \times (d_2^k - d_1^k)}] & 0 & 0 & 0 \\ 0^{d_2^k \times d_1^k} & z_k I_{d_2^k} & [I_{d_2^k}, 0^{d_2^k \times (d_3^k - d_2^k)}] & 0 & \vdots \\ 0 & 0 & z_k I_{d_3^k} & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & [I_{d_{h_k-1}^k}, 0^{d_{h_k-1}^k \times (d_{h_k}^k - d_{h_k-1}^k)}] \\ 0 & 0 & 0 & 0 & z_k I_{d_{h_k}^k} \end{bmatrix} \tag{6}$$

Denote by $d^k = \sum_{i=1}^{h_k} d_i^k$, *then* $J_k \in \mathbb{C}^{d^k \times d^k}$. *The indices* d_i^k *denote the differences of the dimension of the image of* $(J_k - z_k I)^{h_k - i}$ *and the dimension of the image of* $(J_k - z_k I)^{h_k - i+1}$ *for* $i = 1, \dots, h_k$, *where* h_k *denotes the size of the largest Jordan block corresponding to the unit root* z_k.

2. *Let the matrix* $C = [C_1, \dots, C_l, C_{st}]$ *be partitioned according to the partitioning of* J_k. *For each of the matrices* $C_k \in \mathbb{C}^{s \times d^k}, k = 1, \dots, l$ *corresponding to the unit roots* z_k *introduce the following notation: Let* $C_k = [C_k^1, \dots, C_k^{h_k}], C_k^i \in \mathbb{C}^{s \times d_i^k}$. *Further partition* $C_k^i = \left[\begin{array}{cc} C_k^{i,G} & C_k^{i,E} \end{array}\right]$, *with* $C_k^{i,E} \in \mathbb{C}^{s \times (d_i^k - d_{i-1}^k)}$ *and* $C_k^{i,G} \in \mathbb{C}^{s \times d_{i-1}^k}$ *for* $i = 1, \dots, h_k$, *where* $d_0^k = 0$ *is used. Define* $\bar{C}_k^E = [C_k^{1,E}, \dots, C_k^{h_k,E}] \in \mathbb{C}^{s \times d_{h_k}^k}$.
Then $(\bar{C}_k^E)'(\bar{C}_k^E) = I$ *and* $(C_k^{i,G})'(C_k^{j,E}) = 0, j \leq i$ *for* $i = 1, \dots, h_k$ *and* $k = 1, \dots, l$.

3. *Let also* B *be partitioned analogously to* A *and* C, *i.e.* $B = [B_1', \dots, B_l', B_{st}']'$ *with* $B_k = [(B_k^1)', \dots, (B_k^{h_k})']', B_k^i \in \mathbb{C}^{d_i^k \times s}$. *Decompose* $B_k^{h_k} = [(B_k^{h_k,1})', (B_k^{h_k,2})', \dots, (B_k^{h_k,h_k})']'$, $B_k^{h_k,i} \in \mathbb{C}^{(d_i - d_{i-1}) \times s}$. *Every sub-block* $B_k^{h_k,i}$ *for* $i = 1, \dots, h_k$ *is* positive upper triangular. *A matrix* $B \in \mathbb{C}^{c \times s}$, $B = [b_{i,j}]_{i=1,\dots,c, j=1,\dots,s}$ *is called positive upper triangular (p.u.t.), if there exist indices* $1 \leq j_1 < j_2 < \cdots < j_c \leq s$, *such that* $b_{i,j} = 0, j < j_i, b_{i,j_i} > 0$. *I.e.* B *is of the form*

$$\left[\begin{array}{ccccccccc} 0 & \cdots & 0 & b_{1,j_1} & x & & \dots & & x \\ 0 & & \dots & & 0 & b_{2,j_2} & x & \dots & x \\ & & & & & & & & \\ 0 & & & \dots & & & 0 & b_{c,j_c} & x \end{array}\right] \tag{7}$$

with x *here denoting unrestricted entries.*

4. *The state space realization corresponding to the stationary part of the transfer function* (A_{st}, B_{st}, C_{st}) *is represented in a canonical form for stationary state space systems, e.g. in echelon canonical form (cf. Hannan and Deistler, 1988, Theorem 2.5.2).*

5. *For each* $k = 1, \dots, l$ *there exists an index* k' *such that* $\bar{J}_k = J_{k'}$, $\bar{C}_k = C_{k'}$, $\bar{B}_k = B_{k'}$ *and* $\bar{x}_{1,k} = x_{1,k'}$ *for an analogous partitioning of the state.*

The obtained representation has the following properties:

1. *The matrices* $\bar{C}_k^E$ *have full column rank for* $k = 1, \dots, l$. *Hence,* $d_{h_k}^k \leq s$ *and full column rank of* $C_k^{i,E}$ *for* $i = 1, \dots, h_k$ *follow.*

2. *Due to minimality it follows that $B_k^{h_k}$ has full row rank for $k = 1, \dots, l$.*

3. *If also the state is partitioned in the same way as the system matrices, $x_t = [x_{t,1}', \dots, x_{t,l}', x_{t,st}']'$ with $x_{t,k} = [(x_{t,k}^1)', \dots, (x_{t,k}^{h_k})']'$ where $x_{t,k}^i \in \mathbb{C}^{d_i^k}$, then $x_{t,k}^i$ has complex unit root structure $((\omega_k, h_k - i + 1))$. Furthermore $x_{t,k}^i$ is not cointegrated or polynomially cointegrated.*

4. *For each unit root z_k the (complex) integration order h_k of y_t (as well as of x_t) equals the size of the largest Jordan block in J_k.*

PROOF: The existence and the uniqueness of the given representation is stated in Theorem 2 in Bauer and Wagner (2003). Restriction 5 ensures real valuedness of the output process. Properties 1 and 2 are given in Theorem 1 of Bauer and Wagner (2003). In order to see that $x_{t,k}^i$ is indeed integrated of order $h_k - i + 1$ at z_k consider $x_{t+1,k}^i = z_k x_{t,k}^i + [I_{d_i^k}, 0^{d_i^k \times (d_{i+1}^k - d_i^k)}] x_{t,k}^{i+1} + B_k^i \varepsilon_t, i < h_k$ and $x_{t+1,k}^{h_k} = z_k x_{t,k}^{h_k} + B_k^{h_k} \varepsilon_t$. The argument then proceeds recursively: For $i = h_k$, the above equation shows that $x_{t,k}^{h_k}$ is integrated of order $h_k - h_k + 1 = 1$. Since the variance of ε_t is nonsingular and $B_k^{h_k}$ is of full row rank due to minimality (Property 2), no cointegration or polynomial cointegration occurs. Then recursion in $i = h_k - 1, \dots, 1$ proves Property 3. Property 4 finally is a consequence of Properties 1 and 3, using the full column rank of $C_k^1 = C_k^{1,E}$. □

The representation described in the theorem has two convenient features for cointegration analysis. Firstly, the components of the state are decoupled in the sense that they are grouped into blocks of components that are (complex) integrated at exactly one unit root. Secondly, within the blocks corresponding to the different unit roots, the components of the state (i.e. $x_{t,k}^i$) are ordered corresponding to the integration order. This block structure directly shows the chains of state components of increasing integration orders that are relevant for polynomial cointegration. Note that hence e.g. the block of the state corresponding to $z = 1$ is in a triangular representation similar to the representation given in Stock and Watson (1993). Consider

$$y_t = Cx_t + d_t + \varepsilon_t = \sum_{k=1}^{l} \sum_{i=1}^{h_k} C_k^i x_{t,k}^i + C_{st} x_{t,st} + d_t + \varepsilon_t \tag{8}$$

with $C_k^i x_{t,k}^i$ complex integrated of order $h_k - i + 1$ at the unit root z_k, if C_k^i is non-zero. Thus, in $\beta' y_t$ with $\beta \in \mathbb{R}^s$ such that $\beta'[C_k^1, C_k^2, \dots, C_k^j] = 0$ and $\beta' C_k^{j+1} \neq 0$, the order of complex integration corresponding to the unit root z_k is reduced to $h_k - j$. Note that in case that z_k is a member of a pair of complex conjugate unit roots, the vector β from above also reduces the integration order of y_t at $\bar{z}_k$ to $h_k - j$. This follows immediately from restriction 5 and realvaluedness of β, which implies $\overline{\beta' C_k^i} = \beta' \bar{C}_k^i = 0$ for $i = 1, \ldots, j$. The above result shows that the canonical state space representation reveals more information concerning the integration and cointegration properties than contained in the (complex) unit root structure given in Definition 1. This leads us to the definition of a (complex) state space unit root structure.

Definition 5 *The s-dimensional real random process $(y_t; t \in \mathbb{N})$ with minimal state space representation (4) has, using the notation of the above discussion, state space unit root structure $\Omega = \{(\omega_1, (d_1^1, \ldots, d_{h_1}^1)), \ldots, (\omega_l, (d_1^l, \ldots, d_{h_l}^l))\}, 0 < d_i^k \leq d_{i+1}^k$ with $d_{h_k}^k \leq s$ for all $i = 1, \ldots, h_k - 1$ and $k = 1, \ldots, l^{\mathbb{R}}$.*

4 Complex and Real Valued System Representations

The discussion in the previous section and in particular also the formulation of Theorem 1 has been based on complex matrices to simplify the algebra. However, the above results directly lead also to real valued system representations. The key observation in this respect is restriction 5, the fact that for real valued $(y_t; t \in \mathbb{N})$ complex unit roots occur in pairs of complex conjugate roots. For these pairs the corresponding sub-blocks of A, B and C are also complex conjugate. Thus, for example transforming the subsystems (J_k, B_k, C_k), $(\bar{J}_k, \bar{B}_k, \bar{C}_k)$ corresponding to the pair of complex conjugate roots $z_k, \bar{z}_k$ according to

$$J_{k,\mathbb{R}} = \begin{bmatrix} I_{d^k} & I_{d^k} \\ iI_{d^k} & -iI_{d^k} \end{bmatrix} \begin{bmatrix} J_k & 0 \\ 0 & \bar{J}_k \end{bmatrix} \begin{bmatrix} I_{d^k} & I_{d^k} \\ iI_{d^k} & -iI_{d^k} \end{bmatrix}^{-1} = \begin{bmatrix} \mathcal{R}(J_k) & \mathcal{I}(J_k) \\ -\mathcal{I}(J_k) & \mathcal{R}(J_k) \end{bmatrix},$$

$$B_{k,\mathbb{R}} = \begin{bmatrix} I_{d^k} & I_{d^k} \\ iI_{d^k} & -iI_{d^k} \end{bmatrix} \begin{bmatrix} B_k \\ \bar{B}_k \end{bmatrix} = \begin{bmatrix} 2\mathcal{R}(B_k) \\ -2\mathcal{I}(B_k) \end{bmatrix},$$

$$C_{k,\mathbb{R}} = \begin{bmatrix} C_k & \bar{C}_k \end{bmatrix} \begin{bmatrix} I_{d^k} & I_{d^k} \\ iI_{d^k} & -iI_{d^k} \end{bmatrix}^{-1} = \begin{bmatrix} \mathcal{R}(C_k) & \mathcal{I}(C_k) \end{bmatrix},$$

leads to a real valued system representation. Here $\mathcal{R}$ denotes the real part of a complex number and $\mathcal{I}$ the imaginary part.

The focus on only real valued cointegration introduces an asymmetry with respect to the dimensions and structure of the cointegrating spaces corresponding to real and corresponding to complex unit roots. This issue can be most easily exemplified by looking at a process with unit root structure $((\omega_1, 1), \ldots, (\omega_l, 1))$, a process that we call *multiple frequency* I(1) process.

Suppose that $\omega_1 = 0$, thus $z = 1$ is a unit root. For this unit root all cointegrating relationships are given by all vectors $\beta \in \mathbb{R}^s, \beta \neq 0$ such that $\beta' C_1 = 0$. This defines a real space of dimension $s - d^1$, since the matrix C_1 is of full rank in a minimal representation (Property 1 of Theorem 1). This shows the well known relationship between the number of cointegrating relationships $(s - d^1)$, the number of stochastic trends (d^1) and the dimension of the process (s) for I(1) processes.

Let us next consider a complex unit root $z_k = e^{i\omega_k}$ with $0 < \omega_k < \pi$. Note first at this point if we consider also complex cointegration (cf. Remark 1), i.e. if we allow for $\beta \in \mathbb{C}^s$, then the orthogonality constraint $\beta' C_k = 0$, where $C_k \in \mathbb{C}^{s \times d^k}$ say, leads to a $s - d^k$ dimensional complex cointegrating space corresponding to unit root z_k. Thus, the link, discussed above for the unit root $z = 1$, between the number of common cycles and the dimension of the cointegrating space prevails also for the case of complex unit roots when complex cointegration is considered. Restricting attention to only real valued cointegrating vectors breaks this link.

The orthogonality constraint $\beta' C_k = 0$ (solving for $\beta \in \mathbb{R}^s$ this requires orthogonality to both the real and complex part of C_k separately) can be rewritten in real form as $\beta'[\mathcal{R}(C_k), \mathcal{I}(C_k)] = 0$. Full column rank (in $\mathbb{C}^s$) of C_k does not imply full column rank (in $\mathbb{R}^s$) of $[\mathcal{R}(C_k), \mathcal{I}(C_k)]$. The latter matrix can take on any rank from d^k to $\min(2d^k, s)$. Thus, in a real valued discussion there is no link between the number of common cycles and the dimension of the *static* cointegrating space corresponding to a complex unit root of order 1.

As a remark note that the above orthogonality constraint $\beta'[\mathcal{R}(C_k), \mathcal{I}(C_k)] = 0$ is exactly the same orthogonality constraint as the one that arises from the real valued system representation, when the sub-blocks (J_k, B_k, C_k) and $(\bar{J}_k, \bar{B}_k, \bar{C}_k)$ are transformed to a real valued sub-system comprising both blocks, see (9). In the corresponding real valued system representation, the block in the real C-matrix corresponding to the pair of complex conjugate unit roots, $C_{k,\mathbb{R}}$ say again, is given by $[\mathcal{R}(C_k), \mathcal{I}(C_k)]$.

In the case of real unit roots, minimum degree polynomial vectors are restricted to be constant in the MFI(1) case. However in the complex unit root case, the elementary difference filter $\Delta_\omega(z)$ is of polynomial degree two and therefore minimum degree polynomial cointegrating vectors of polynomial degree one might exist. Therefore the focus on only real valued cointegration gives rise to *dynamic* cointegrating relationships for MFI(1) processes, see e.g. Johansen and Schaumburg (1999). These are cointegrating relationships of polynomial degree 1, $\beta(z) = \beta_0 + \beta_1 z$ with $\beta_0, \beta_1 \in \mathbb{R}^s$. Consider only the contribution of $C_k x_{t,k}$ to the output y_t to observe

$$(\beta_0' + \beta_1' z) C_k B_k z_t = \beta_0' C_k B_k \varepsilon_{t-1} + [\beta_0' C_k z_k + \beta_1' C_k] \, B_k \sum_{j=1}^{t-2} z_k^{j-1} \varepsilon_{t-j-1}.$$

with $z_t = \sum_{j=1}^{t-1} z_k^{j-1} \varepsilon_{t-j}$. Thus dynamic cointegration at the unit root z_k occurs for

$$\left[\begin{array}{cc} \beta_0' & \beta_1' \end{array} \right] \left[\begin{array}{c} C_k z_k \\ C_k \end{array} \right] = 0. \tag{9}$$

The dynamic cointegrating relationships are found via orthogonality relationships over a real space of dimension $2s$ by equating the real and the imaginary part of (9) separately. Note that equivalently again the real valued system representation can be taken to recover the dynamic cointegrating spaces from

$$\left[\begin{array}{cc} \beta_0' & \beta_1' \end{array} \right] \left[\begin{array}{c} C_{k,\mathbb{R}} J_{k,\mathbb{R}} \\ C_{k,\mathbb{R}} \end{array} \right] = \left[\begin{array}{cc} \beta_0' & \beta_1' \end{array} \right] \left[\begin{array}{cc} \cos\omega_k \mathcal{R}(C_k) - \sin\omega_k \mathcal{I}(C_k) & \sin\omega_k \mathcal{I}(C_k) + \cos\omega_k \mathcal{R}(C_k) \\ \mathcal{R}(C_k) & \mathcal{I}(C_k) \end{array} \right] = 0 \tag{10}$$

The matrix in equation (10) can be shown to have full column rank. Thus, the dynamic cointegrating spaces are seen to be of dimension $2(s - d^k)$ for the complex unit roots in MFI(1) processes. This reestablishes the analogy to the well known relation for the unit roots ± 1. Note that in this space the static cointegrating relationships are contained as a subset with $\beta_1 = 0$.

The above discussion makes clear that in the case of complex unit roots and when considering real cointegration it suffices to investigate the system blocks in the complex representation corresponding to the unit roots with frequencies in the interval $[0, \pi]$. This, as has been illustrated in the example and holds true in general, is equivalent to consider the real valued blocks corresponding to pairs of complex conjugate unit roots.

5 An I(2) Example

Let us next illustrate with a small example the fact that the developed representation reveals the polynomial cointegrating spaces clearly in systems with higher integration orders. In the example we consider only the unit root $z = 1$ and neglect other nonstationary components as well as stationary dynamics for simplicity. Let the state space unit root structure for the example be given by $\Omega = \{(0, (1, 2))\}$. Thus, combining the definition of the state space unit root structure with Theorem 1 above, it follows that the output is integrated of order 2 at this unit root. This also becomes clear immediately from the corresponding state equation:

$$\begin{bmatrix} x^1_{t+1,1} \\ x^2_{t+1,1} \end{bmatrix} = \left[\begin{array}{c|cc} 1 & 1 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \begin{bmatrix} x^1_{t,1} \\ x^2_{t,1} \end{bmatrix} + \left[\begin{array}{c} B^1_1 \\ \hline B^2_1 \end{array}\right] \varepsilon_t,$$

$$y_t = \left[\begin{array}{c|c} C^1_1 & C^2_1 \end{array}\right] x_t + \varepsilon_t$$

with $x^1_{t,1} \in \mathbb{R}$, $x^2_{t,1} \in \mathbb{R}^2$, $B^1_1 \in \mathbb{R}^{1\times s}$, $B^2_1 \in \mathbb{R}^{2\times s}$, $C^1_1 \in \mathbb{R}^{s\times 1}$ and $C^2_1 = \begin{bmatrix} C^{2,G}_1 & C^{2,E}_1 \end{bmatrix} \in \mathbb{R}^{s\times 2}$ assuming again that $y_t, \varepsilon_t \in \mathbb{R}^s$. We assume that x_1 is chosen such that $\Delta^2 y_t$ is stationary. From the theorem above we know that in a minimal representation there is no cointegration among the components of $x^2_{t,1}$, i.e. these two components are two linearly independent I(1) variables. It is also clear, both from the theorem and the above equations, that $\Delta x^1_{t+1,1}$ is equal to the first component of $x^1_{t+1,2}$, denoted by $x^{2,G}_{t,1}$ henceforth, plus $B^1_1\varepsilon_t$. It is this type of recursive integrating relationship between components of the state corresponding to the same Jordan chain that drives polynomial cointegration, as will be made clear below. The fact that the developed representation directly reveals these chains is the reason for rendering the representation particularly suited for studying polynomial cointegration.

It will be shown below that orthogonality relationships, given that the polynomial vector $\beta(z)$ is parameterized in powers of the filter $(1 - z)$ for this example with the unit root at $z = 1$ recover the polynomial cointegrating spaces. Hence for the I(2) case, let $\beta(z) = \tilde{\beta}_0 + \tilde{\beta}_1\Delta$. From Definition 4 and the discussion in the previous section it is clear that for the I(2) case the maximum required polynomial degree is equal to one. Thus, for $\beta(z)'y_t$

we obtain

$$\begin{aligned}\beta(z)'y_t &= \tilde{\beta}_0'C_1^1x_{t,1}^1 + \tilde{\beta}_0'C_1^{2,G}x_{t,1}^{2,G} + \tilde{\beta}_0'C_1^{2,E}x_{t,1}^{2,E} + \tilde{\beta}_1'C_1^1\Delta x_{t,1}^1 \\ &\quad + \tilde{\beta}_1'C_1^{2,G}\Delta x_{t,1}^{2,G} + \tilde{\beta}_1'C_1^{2,E}\Delta x_{t,1}^{2,E} + \beta(z)'\varepsilon_t.\end{aligned}$$

Note next that $\tilde{\beta}_1'C_1^{2,G}\Delta x_{t,1}^{2,G}$ and $\tilde{\beta}_1'C_1^{2,E}\Delta x_{t,1}^{2,E}$ are stationary and that $\Delta x_{t,1}^1$ is, as already noted above, up to stationary terms equal to $x_{t,1}^{2,G}$. Thus, the nonstationary components of the expression above are given by:

$$\tilde{\beta}_0'C_1^1x_{t,1}^1 + (\tilde{\beta}_0'C_1^{2,G} + \tilde{\beta}_1'C_1^1)x_{t,1}^{2,G} + \tilde{\beta}_0'C_1^{2,E}x_{t,1}^{2,E}.$$

Here $\tilde{\beta}_0'C_1^1x_{t,1}^1$ has unit root structure $((0,2))$, if $\tilde{\beta}_0'C_1^1 \neq 0$. Otherwise, when $\tilde{\beta}_0'C_1^1 = 0$, $\beta(z)'y_t$ is integrated of order 1 at the unit root $z = 1$ or is not integrated. Since $x_{t,1}^{2,G}$ and $x_{t,1}^{2,E}$ are not cointegrated (Property 3 of Theorem 1), $\beta(z)'y_t$ is stationary if and only if additionally to $\tilde{\beta}_0'C_1^1 = 0$ also $\tilde{\beta}_0'C_1^{2,G} + \tilde{\beta}_1'C_1^1 = 0$ and $\tilde{\beta}_0'C_1^{2,E} = 0$ hold. This implies that all cointegrating relationships can be uncovered from the following orthogonality conditions:

$$\begin{bmatrix} \tilde{\beta}_0' & \tilde{\beta}_1' \end{bmatrix} \left[\begin{array}{c|cc} C_1^1 & C_1^{2,G} & C_1^{2,E} \\ 0 & C_1^1 & 0 \end{array}\right] = 0. \tag{11}$$

Setting $\tilde{\beta}_1 = 0$, this condition reveals the necessary orthogonality restrictions for the existence of static cointegration: If orthogonality holds only for the first block-column (separated by the vertical line), the order of integration is reduced from 2 to 1, if orthogonality holds with respect to all three block-columns of the first block-row, the transformed process $\tilde{\beta}_0'y_t$ is stationary. Furthermore looking at the first block-column it is e.g. obvious that every polynomial of the form $0 + \tilde{\beta}_1(1-z)$ reduces the integration order by (at least) one. These trivial relationships are, however, not interesting and the representation in terms of block-orthogonality constraints in connection with the reparameterization in terms of powers of Δ directly allows to distinguish the *relevant* from the trivial relationships. Rewriting polynomial cointegration in terms of the observations y_t, it is a linear combination of y_t and Δy_t (or equivalently between y_t and y_{t-1}) that is, in this I(2) example, stationary. In the appendix the representation discussed here for a small example is analyzed for any given unit root and arbitrary integration order.

6 Polynomial Cointegrating Relationships

The representation presented in Section 3 leads directly to necessary conditions for polynomial cointegrating relationships when investigating all unit roots jointly. It follows from the state space equations (4) that for each $t > h$ the output can be written as (setting $d_t = 0$ for notational simplicity)

$$y_t = Cx_t + \varepsilon_t = C(Ax_{t-1} + B\varepsilon_{t-1}) + \varepsilon_t = \cdots = CA^h x_{t-h} + \sum_{j=0}^{h-1} CA^j B\varepsilon_{t-j-1} + \varepsilon_t.$$

Let $Y_{t,q}^- = [y_t', y_{t-1}', \ldots, y_{t-q}']'$ for $t > q$ denote the vector of stacked outputs. Consider the equation given above stacked for all outputs contained in $Y_{t,q}^-$, where h is adopted to give the same state x_{t-q} in all equations.

$$Y_{t,q}^- = \underbrace{\begin{bmatrix} C_u A_u^q \\ C_u A_u^{q-1} \\ \vdots \\ C_u \end{bmatrix}}_{\tilde{\mathcal{O}}_{q,u}} x_{t-q,u} + \underbrace{\begin{bmatrix} C_{st} A_{st}^q \\ C_{st} A_{st}^{q-1} \\ \vdots \\ C_{st} \end{bmatrix}}_{\tilde{\mathcal{O}}_{q,st}} x_{t-q,st} + \underbrace{\begin{bmatrix} I & CB & \ldots & CA^{q-2}B & CA^{q-1}B \\ 0 & I & CB & \ldots & CA^{q-2}B \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & CB \\ 0 & \ldots & \ldots & 0 & I \end{bmatrix}}_{\mathcal{E}_q} \begin{bmatrix} \varepsilon_t \\ \varepsilon_{t-1} \\ \vdots \\ \varepsilon_{t-q} \end{bmatrix} \tag{12}$$

where the subscript u refers to quantities corresponding to the unit roots and the subscript st refers to quantities corresponding to the stationary part of the system. This can be used in

$$\beta(z)'y_t = \sum_{j=0}^{q} \beta_j' y_{t-j} = \underbrace{\begin{bmatrix} \beta_0' & \beta_1' & \ldots & \beta_q' \end{bmatrix}}_{\vec{\beta}'} Y_{t,q}^- = \vec{\beta}' \left(\tilde{\mathcal{O}}_{q,u} x_{t-q,u} + \tilde{\mathcal{O}}_{q,st} x_{t-q,st} + \mathcal{E}_q E_{t,q}^- \right)$$

where $\mathcal{E}_q$ denotes the matrix in (12) pre-multiplying the contribution of the noise, $E_{t,q}^- = [\varepsilon_t', \varepsilon_{t-1}', \ldots, \varepsilon_{t-q}']'$ and $\vec{\beta} = [\beta_0', \beta_1', \ldots, \beta_q']'$. Clearly the latter two terms are stationary for suitable initial value $x_{1,st}$ such that $x_{t,st}$ is stationary and $\vec{\beta}'\mathcal{E}_q E_{t,q}^-, t > q$ is an MA process. Hence the unit root structure of the process $(\beta(z)'y_t; t \in \mathbb{N})$ is solely determined by the respective properties of $\vec{\beta}'\tilde{\mathcal{O}}_{q,u} x_{t-q,u}$. The key to understand the cointegration properties of this process is the discussed decomposition of the state x_t: It has been stated in Theorem 1 that the state can be partitioned into blocks $x_{t,k}^i$ that are complex integrated of orders $h_k - i + 1$ only at one unit root z_k. Furthermore each vector process $x_{t,k}^i$ is not cointegrated

(Property 3 of Theorem 1). Thus, it follows that the polynomial cointegrating vectors of the different orders can be found via orthogonality to certain sub-blocks of the matrix $\tilde{\mathcal{O}}_{q,u}$. Partition therefore $\tilde{\mathcal{O}}_{q,u} = [\tilde{\mathcal{O}}^i_{q,k}]_{k=1,\ldots,l,i=1,\ldots,h_k}$ with $\tilde{\mathcal{O}}^i_{q,k} \in \mathbb{C}^{(q+1)s\times d^k_i}$, i.e the partitioning is performed according to the decomposition of the nonstationary components of x_t into $x^i_{t,k}$ and a potential third subscript u is omitted.

Our focus on real cointegration implies that it is sufficient to consider only the blocks corresponding to $z_k = e^{i\omega_k}$ such that $0 \leq \omega_k \leq \pi$, compare also the discussion in section 4. This follows from the fact that for real valued polynomial cointegrating vectors $\vec{\beta}'\tilde{\mathcal{O}}^i_{q,k} = 0$ is equivalent to $\vec{\beta}'\tilde{\mathcal{O}}^i_{q,k'} = 0$ for k' such that $z_k = \bar{z}_{k'}$. It is equivalently possible to base the analysis on the real valued matrices $\tilde{\mathcal{O}}^i_{\mathbb{R},q,k} \in \mathbb{R}^{(q+1)s\times 2d^k_i}, z_k \neq \pm 1$. We formulate the theorems using the complex representation and consider the real valued representation in Section 8 in an example.

The next result shows that the matrices $\tilde{\mathcal{O}}^i_{q,k}$ play a fundamental role in the determination of polynomial cointegrating relationships. We also show that it is sufficient to consider only the polynomial degree $q = n-1$, since we are only concerned with the unit root structure of the transformed process $\beta(z)'y_t$ and disregard for the moment the issues of non-triviality and minimum degree of the polynomial $\beta(z)$.

Theorem 2 *Let $(y_t; t \in \mathbb{N})$ be as in Theorem 1. Then $(\beta(z)'y_t; t \in \mathbb{N})$ is of unit root structure $((\omega_1, h^p_1), \ldots, (\omega_{l^{\mathbb{R}}}, h^p_{l^{\mathbb{R}}}))$, if and only if $\beta(z) = \sum_{j=0}^q \beta_j z^j, \beta_j \in \mathbb{R}^s$ is such that $\vec{\beta}'\tilde{\mathcal{O}}^i_{q,k} = 0$ for $i = 0, \ldots, h_k - h^p_k, k = 1, \ldots, l^{\mathbb{R}}$ and $\vec{\beta}'\tilde{\mathcal{O}}^{h_k - h^p_k+1}_{q,k} \neq 0$ for $k = 1, \ldots, l^{\mathbb{R}}$ such that $h^p_k > 0$. Here $\tilde{\mathcal{O}}^0_{q,k} = 0$.*

There exists a vector $\beta(z) = \sum_{j=0}^q \beta_j z^j$ such that $(\beta(z)'y_t; t \in \mathbb{N})$ is of the given unit root structure, if and only if there exists a vector $\beta_{n-1}(z) = \sum_{j=0}^{n-1} \beta_{j,n-1} z^j$ such that $(\beta_{n-1}(z)'y_t; t \in \mathbb{N})$ is of the given unit root structure.

If $\vec{\beta}'\tilde{\mathcal{O}}_{q,u} = 0$, then there exists a linearly deterministic term d_t such that $(\beta(z)'(y_t - d_t); t \in \mathbb{N})$ is stationary.

PROOF: The representation

$$\beta(z)'y_t = \vec{\beta}'\tilde{\mathcal{O}}_{q,u}x_{t-q,u} + \nu_t = \sum_{k=1}^{l}\sum_{i=1}^{h_k} \vec{\beta}'\tilde{\mathcal{O}}^i_{q,k}x^i_{t-q,k} + \nu_t$$

and the properties of the state components $x^i_{t,k}$ as discussed above, directly imply that if $\vec{\beta} \in \mathbb{R}^{s(q+1)}$ is as described in the theorem, then the process $(\beta(z)'y_t; t \in \mathbb{N})$ has the unit root structure given in the theorem. The term ν_t collects all stationary and linearly deterministic contributions. Conversely suppose that there exists a vector polynomial $\beta(z) = \sum_{j=0}^{q} \beta_j z^j$ such that $(\beta(z)'y_t; t \in \mathbb{N})$ has the given unit root structure. Then the representation given above implies that the corresponding vector $\vec{\beta}$ has to fulfil the orthogonality restrictions stated in the theorem.

In order to prove the second claim, the sufficiency of considering $q = n-1$, three cases concerning q have to be considered: The case $q = n-1$ is trivial. In case that $q < n-1$ setting $\beta_{j,n-1} = 0, q < j < n$ proves the result. If $q > n-1$ the result follows from the fact that each matrix satisfies its characteristic equation, and hence $A^q, q \geq n$ can be written as a linear combination of $I_n, A, \ldots, A^{n-1}$. Consider e.g. $A^n = \alpha_0 I_n + \alpha_1 A^1 + \cdots + \alpha_{n-1} A^{n-1}$. Then $\sum_{j=0}^{n} \beta_j' C A^j =$

$$\sum_{j=0}^{n-1} \beta_j' C A^j + \beta_n' C A^n = \sum_{j=0}^{n-1} \beta_j' C A^j + \beta_n' C(\alpha_0 I_n + \alpha_1 A^1 + \cdots + \alpha_{n-1} A^{n-1}) = \sum_{j=0}^{n-1} (\beta_j + \alpha_j \beta_n)' C A^j$$

Completely analogously A^q for $q > n$ can be dealt with. This proves the second statement of the theorem.

The third statement follows from equation (12): For $t > q$ this equation shows that for $\vec{\beta}'\tilde{\mathcal{O}}_{q,u} = 0$ it follows that $\beta(z)'y_t = \vec{\beta}'\tilde{\mathcal{O}}_{q,st} x_{t-q,st} + \vec{\beta}'\mathcal{E}_q E_{t,q}^-$ for $t > q$. $x_{t-q,st}$ is (possibly up to a linearly deterministic process contained in $C_{st} A_{st}^{t-q-1} x_{1,st}$) stationary. Thus, the process $(\beta(z)'y_t; t \in \mathbb{N})$ is for $t > q$ up to a linearly deterministic process stationary. As mentioned in Remark 2, a process composed of the observations of $\beta(z)'y_t$ for $t \leq q$ and 0 afterwards is linearly deterministic. This proves the theorem. $\square$

The theorem shows that the integration properties of $(\beta(z)'y_t; t \in \mathbb{N})$ can be easily determined using some simple orthogonality properties. These properties directly emerge from the canonical state space representation developed in Bauer and Wagner (2003) and are shown here to be suitable for polynomial cointegration analysis as well. According to our opinion the partitioning of the state into the blocks $x_{t,k}^i$ is very useful in understanding the cointegration and in particular also the polynomial cointegration properties of $(y_t; t \in \mathbb{N})$, especially for processes with a complicated unit root structure. Neither in a Wold type representation nor in an ARMA representation are the underlying cointegrating relationships so clearly seen and interpretable.

The above Theorem 2 gives a classification of the integration properties of the transformed output process $(\beta(z)'y_t; t \in \mathbb{N})$. The orthogonality condition is only necessary for polynomial cointegration, but not sufficient. This is clear since e.g. the null vector always satisfies every orthogonality constraint. Sufficient conditions for polynomial cointegration, which requires to find non-trivial polynomial cointegrating relationships, are the topic of the following section.

7 Existence of Polynomial Cointegrating Relationships

The previous section provided conditions for polynomial vectors to reduce the order of integration. In order to qualify for a polynomial cointegrating relationship, the polynomial additionally has to be non-trivial. Remember that this is characterized (cf. Definition 3) by the fact that the polynomial has to be non-zero for at least one unit root where the integration order is reduced and by $\beta(0) \neq 0$. It is clear that this condition is not related to the orthogonality property of Theorem 2. Focus again at one unit root, $z_k = e^{i\omega_k}$ say, and consider

$$\beta(z_k) = \sum_{j=0}^{n-1} \beta_j z_k^j = \begin{bmatrix} I_s & z_k I_s & \dots & z_k^{n-1} I_s \end{bmatrix} \vec{\beta} = Q_{n-1}(z_k)\vec{\beta}$$

where the last equality defines the operator $Q_{n-1}(z_k)$. Hence the restriction $\beta(z_k) \neq 0$ is equivalent to $Q_{n-1}(z_k)\vec{\beta} \neq 0$. Note that for complex unit roots, the matrix $Q_{n-1}(z_k)$ is complex valued. This inequality condition can be combined with Theorem 2 to obtain a characterization of the existence of non-trivial polynomial cointegrating relationships.

Theorem 3 *Let $(y_t; t \in \mathbb{N})$ be as in Theorem 1. Furthermore let $P = ((\omega_1, h_1, h_1^p), \dots, (\omega_{l^{\mathbb{R}}}, h_{l^{\mathbb{R}}}, h_{l^{\mathbb{R}}}^p))$ and $O(P)$ denote the matrix, whose block-columns consist of $\{\tilde{\mathcal{O}}_{n-1,k}^i, i = 1, \dots, h_k - h_k^p, k = 1, \dots, l^{\mathbb{R}}\}$. Then $(y_t; t \in \mathbb{N})$ is polynomially cointegrated of order $((\omega_1, h_1, h_1^p), \dots, (\omega_{l^{\mathbb{R}}}, h_{l^{\mathbb{R}}}, h_{l^{\mathbb{R}}}^p))$, if and only if there exists a vector $b \in \mathbb{R}^{ns}$ such that (using $\tilde{\mathcal{O}}_{n-1,k}^{h_k+1} = I_{ns}$)*

$$\begin{array}{lcll} O(P)'b & = & 0, & \\ (\tilde{\mathcal{O}}_{n-1,k}^{h_k - h_k^p + 1})'b & \neq & 0, & k = 1, \dots, l^{\mathbb{R}}, \\ Q_{n-1}(z_k)b & \neq & 0, & \text{for some } k \in \{1, \dots, l^{\mathbb{R}}\} \text{ such that } h_k - h_k^p > 0. \end{array} \quad (13)$$

Let Π_O denote the orthogonal projection onto the space spanned by the columns of $O(P)$ and let $\Pi_{O^\perp} = I_{ns} - \Pi_O$. Then fulfillment of (13) is equivalent to the existence of one

$k \in \{1, \ldots, l^{\mathbb{R}}\}$ *such that* $h_k - h_k^p > 0$ *and*

$$\|Q_{n-1}(z_k)\Pi_{O^\perp}\|_{Fr} \prod_{j=1}^{l^{\mathbb{R}}} \|\Pi_{O^\perp} \tilde{\mathcal{O}}_{n-1,j}^{h_j - h_j^p + 1}\|_{Fr} \neq 0 \tag{14}$$

where $\|.\|_{Fr}$ *denotes the Frobenius norm.*

PROOF: The first statement follows immediately from Theorem 2, implying the existence of a vector polynomial that reduces the integration order, and the definition of $Q_{n-1}(z_k)$ ascertaining the non-triviality of this vector polynomial.

In order to show equivalence of the two statements the argument proceeds indirectly: Assume that there exists no vector $b \in \mathbb{R}^{ns}$ for which both $O(P)'b = 0$ and the two inequality constraints in (13) hold: Then either the space spanned by the columns of $\tilde{\mathcal{O}}_{n-1,j}^{h_j - h_j^p + 1}$ for some j or the space spanned by the columns of $Q_{n-1}(z_k)'$ (or both) is a subspace of $O(P)$ and hence the expression (14) of the theorem is equal to zero.

On the other hand, if there exists a vector b fulfilling (13) it follows that $b = \Pi_{O^\perp} b$ and hence $b'\tilde{\mathcal{O}}_{n-1,k}^{h_k - h_k^p + 1} = b'\Pi_{O^\perp}\tilde{\mathcal{O}}_{n-1,k}^{h_k - h_k^p + 1}$ is nonzero. Therefore $\Pi_{O^\perp}\tilde{\mathcal{O}}_{n-1,k}^{h_k - h_k^p + 1}$ is nonzero and consequently also its Frobenius norm is nonzero. From an analogous argument also $\Pi_{O^\perp} Q_{n-1}(z_k)' \neq 0$ follows, which proves the conjecture. □

Remark 4 *Theorems 2 and 3 can completely analogously be used for considering complex polynomial cointegration. The only differences are that* $b \in \mathbb{C}^{ns}$ *has to be considered and that the index set for* k *is given by* $k = 1, \ldots, l$*. This allows to disentangle the complex cointegrating spaces to each of the members of pairs of complex conjugate unit roots.*

The preceding theorem gives an expression for the existence of non-trivial polynomial cointegrating relations of polynomial degree $n-1$. It is straightforward to verify, that the theorem also gives a characterization of the existence of non-trivial polynomial cointegrating relations of polynomial degree q for arbitrary $q \in \mathbb{N}$ by simply replacing q for $n-1$ in the expressions.

The concept of minimum degreeness is not contained in the analysis up to now. Recall the definition of non-minimum degree polynomial cointegration: A vector polynomial $\beta(z)$ of degree q, say, is non-minimum degree, if $\beta(z) = \sum_{j=1}^m p_j(z)\beta_j(z)$, where $p_j(z)\beta_j(z)$ reduce the unit root structure at least as much as $\beta(z)$ and certain restrictions on the polynomial degree of $\beta_j(z)$ and $p_j(z)\beta_j(z)$ hold (compare Definition 4). Reformulate this condition in terms of the vectors $\vec{\beta}$: Let $\mathrm{vec}[\gamma(z)] = [\gamma_0', \gamma_1', \dots, \gamma_q']' \in \mathbb{R}^{(q+1)s}$ denote the vector of stacked polynomial coefficients of $\gamma(z) = \sum_{j=0}^{q} \gamma_j z^j$ for each vector polynomial $\gamma(z)$ of maximal degree q. Then non-minimum degreeness is equivalent to $\mathrm{vec}[\beta(z)] \in \mathrm{span}\{\mathrm{vec}[\beta_j(z)z^l], j = 1, \dots, m, l = 0, \dots, q-1-\deg(\beta_j(z))\}$, where deg denotes the polynomial degree of a vector polynomial. If $\beta(z)$ is minimum-degree, $\beta(z)$ is not contained in this set. It follows that the orthogonal projection of $\mathrm{vec}[\beta(z)]$ onto the orthogonal complement of the above mentioned space, say $\tilde{\beta}(z)$, is of degree q. Hence the existence of a minimum degree cointegrating polynomial of any given polynomial degree is equivalent to the existence of a minimum degree cointegrating polynomial of the same degree, which moreover is orthogonal to all polynomial cointegrating vectors $\mathrm{vec}[z^l\beta_j(z)]$ reducing the unit root structure at least as much as $\beta(z)$ does. This latter orthogonality constraint adds new columns to $O(P)$. Using a recursive procedure in the polynomial degree of $\beta(z)$ the existence of minimum degree non-trivial polynomial cointegrating relations of any given polynomial degree can be characterized using expression (14), where the definition of $\Pi_{O^\perp}$ now includes also lower order polynomial cointegrating relations. We will not go into more detail in this respect.

8 An Illustrative Example

To illustrate the results of the preceding sections we now analyze a small example. We consider a 3-dimensional process with unit roots at 1 and $\pm i$ with *complex* state space unit root structure $\{(0,(1,2)),(\frac{\pi}{2},(1,1)),(\frac{3\pi}{2},(1,1))\}$. The integration orders corresponding to each of the unit roots are equal to 2. For simplicity stationary dynamics are neglected in the discussion:

$$y_t = C_0 x_{t,0} + C_{\pi/2} x_{t,\pi/2} + C_{3\pi/2} x_{t,3\pi/2} + \varepsilon_t$$

where notation is chosen here to use the unit root frequency as subscript. Due to realvaluedness of y_t only one of the blocks corresponding to the pair of complex unit roots $\pm i$ has to be investigated. We will equivalently investigate the real block $C_{\pi/2,\mathbb{R}}$ corresponding to the pair of unit roots $\pm i$ below. The blocks of the state equation are given by:

$$x_{t+1,0} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x_{t,0} + \begin{bmatrix} B_0^1 \\ B_0^{2,G} \\ B_0^{2,E} \end{bmatrix} \varepsilon_t$$

and

$$x_{t+1,\pi/2} = \begin{bmatrix} i & 1 \\ 0 & i \end{bmatrix} x_{t,\pi/2} + \begin{bmatrix} B_{\pi/2}^1 \\ B_{\pi/2}^{2,G} \end{bmatrix} \varepsilon_t$$

respectively. Let the matrix C_0 be given in canonical form as described in Theorem 1 by:

$$C_0 = \begin{bmatrix} C_0^1 & C_0^{2,G} & C_0^{2,E} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/\sqrt{2} \\ 0 & -1/2 & 1/\sqrt{2} \end{bmatrix}.$$

It is easily verified by straightforward computations that this matrix indeed fulfills all restrictions formulated in Theorem 1. Furthermore note that this matrix is nonsingular. Let the block corresponding to the unit root $z = i$ be given (in complex canonical form) by:

$$C_{\pi/2} = \begin{bmatrix} C_{\pi/2}^1 & C_{\pi/2}^{2,G} \end{bmatrix} = \begin{bmatrix} 0 & i \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since the matrix $\tilde{\mathcal{O}}_q$ contains only the matrices A and C, there is no need to specify a B-matrix for this example. The real valued canonical representation for the system matrices A and C is given by (numbers rounded):

$$A_{\mathbb{R}} = \left[\begin{array}{ccc|cc|cc} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{array}\right], \quad C_{\mathbb{R}} = \left[\begin{array}{ccc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0.5 & 0.707 & 1 & 0 & 0 & 0 \\ 0 & -0.5 & 0.707 & 0 & 1 & 0 & 0 \end{array}\right]$$

Following the discussion of Theorem 2 the matrix investigated is $\tilde{\mathcal{O}}_{\mathbb{R},6}$. As discussed we know that most of the polynomial cointegrating relationships recovered via orthogonality to this matrix are not be of minimum degree and are potentially also trivial. Especially when the analysis is focused on one unit root only it is sufficient to consider smaller matrices such as $\tilde{\mathcal{O}}_{\mathbb{R},3}$. Investigating the full matrix once is however an interesting exercise that clarifies the content of Theorem 2 and 3. In $\tilde{\mathcal{O}}_{\mathbb{R},6}$ below, the partitioning already indicates the sub-blocks $\tilde{\mathcal{O}}^{i}_{\mathbb{R},6,k}$:

$$\tilde{\mathcal{O}}_{\mathbb{R},6} = \left[\begin{array}{c|cc||cc|cc}
1 & 6 & 0 & 0 & 0 & 0 & -1 \\
0 & 0.5 & 0.7071 & -1 & 0 & 0 & 6 \\
0 & -0.5 & 0.7071 & 0 & -1 & 0 & 0 \\
\hline
1 & 5 & 0 & 0 & -1 & 0 & 0 \\
0 & 0.5 & 0.7071 & 0 & 5 & 1 & 0 \\
0 & -0.5 & 0.7071 & 0 & 0 & 0 & 1 \\
\hline
1 & 4 & 0 & 0 & 0 & 0 & 1 \\
0 & 0.5 & 0.7071 & 1 & 0 & 0 & -4 \\
0 & -0.5 & 0.7071 & 0 & 1 & 0 & 0 \\
\hline
1 & 3 & 0 & 0 & 1 & 0 & 0 \\
0 & 0.5 & 0.7071 & 0 & -3 & -1 & 0 \\
0 & -0.5 & 0.7071 & 0 & 0 & 0 & -1 \\
\hline
1 & 2 & 0 & 0 & 0 & 0 & -1 \\
0 & 0.5 & 0.7071 & -1 & 0 & 0 & 2 \\
0 & -0.5 & 0.7071 & 0 & -1 & 0 & 0 \\
\hline
1 & 1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0.5 & 0.7071 & 0 & 1 & 1 & 0 \\
0 & -0.5 & 0.7071 & 0 & 0 & 0 & 1 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0.5 & 0.7071 & 1 & 0 & 0 & 0 \\
0 & -0.5 & 0.7071 & 0 & 1 & 0 & 0
\end{array}\right]$$

One note of caution is in order here: Since we use the real valued canonical form, the columns corresponding to double integration at $\pm i$ are contained in the fourth and sixth column of this matrix and not, as one might think, in columns four and five. This follows, since in the complex representation columns four and six correspond to the double integration, and the real valued representation simply separates real and imaginary parts, but does not change with the ordering of the columns. Alternatively a real-valued representation taking the ordering analogously to the case of unit roots at ± 1 could be developed.

Let us start by investigating the polynomials of degree 0, i.e. the static cointegrating relationships. Thus only the first three rows of $\tilde{\mathcal{O}}_{\mathbb{R},6}$ have to be considered. This block-row is given by CA^6 and non-singularity of A shows that orthogonality to certain columns of C is necessary and sufficient to derive the static cointegrating relationships. We start with considering the unit root $z = 1$. Any vector $\beta_0 = [0, b_2, b_3]'$ implies that $\beta_0' y_t$ is integrated of order one at $z = 1$, if it is at all integrated. Since the matrix C_0 has full column-rank

there exists no vector $\beta_0 \neq 0$ that reduces the integration order to zero at the unit root $z = 1$. Corresponding to the pair of unit roots $z = \pm i$, one obtains that every vector of the form $\beta_0 = [b_1, 0, b_3]'$ reduces the integration order at $z = \pm i$ from two to one, but no linear combination of the components of y_t exists, such that the pair of unit roots is eliminated.[1]
Let us next investigate polynomial cointegrating relationships, noting once again that the orthogonality to certain blocks of $\tilde{\mathcal{O}}^i_{\mathbb{R},6,k}$ includes both trivial and in particular also non-minimum degree relationships. Let us start with the unit root $z = 1$ again. For this unit root we find via orthogonality to the first column of $\tilde{\mathcal{O}}_{\mathbb{R},6}$ all polynomial relationships that reduce the integration order from 2 to 1 (or to 0, if in addition orthogonality to the second block-column prevails). Note that of course no *relevant* nonconstant polynomial cointegrating relationships that reduce the integration order from 2 to exactly 1 at the unit root $z = 1$ can exist, when one is considering this unit root only. However, as we will see below, some of the relationships found might be *non-trivially* cointegrating at the other unit roots, which in turn may lead them to be classified as *non-trivial*, compare Definitions 2 and 3.
The ortho-complement of the first column of $\tilde{\mathcal{O}}_{\mathbb{R},6,k}$ is a space of dimension 20, being the orthogonal complement of a vector in $\mathbb{R}^{21}$. 14 basis vectors of this space are of the form $\beta_i = [0, b_2, b_3]', i = 0, \dots, 6$ for arbitrary $b_2, b_3 \in \mathbb{R}$. Twelve of them are of non-minimum degree, being lagged versions of the static cointegrating relationships. The remaining 6 basis vectors are given by $\beta_i = [b_1, 0, 0]', \beta_{i+1} = [-b_1, 0, 0]', i = 0, \dots, 5$. These are lagged versions of the first difference operator applied to the first component of y_t, which is the only component of y_t that is integrated of order 2 at $z = 1$. These 6 relationships all fulfill $\beta(1) = 0$ and five of them are non-minimum degree. Nevertheless the polynomial $[\Delta, -1, 1]'$ is non-trivial, since it also induces a reduction of the integration orders at $\pm i$ from 2 to 1 whilst fulfilling $\beta(i) \neq 0$. Thus, this example shows that in order to find all polynomial cointegrating relationships it is not sufficient to find all non-trivial polynomial cointegrating relationships for each of the unit roots in turn, but that all unit roots have to be considered jointly.
Let us next add the second block-column to investigate the polynomial cointegrating relationships that wipe out the nonstationarity at frequency 0. This requires finding a basis

[1] If one allows for complex cointegrating relationships, there exist relationships that reduce the complex integration order at one of the complex roots to zero. For example $\beta_0 = [i, 0, 1]'$ results in a complex process $\beta_0' y_t$ which is not complex integrated at the unit root i.

for an 18-dimensional space, which is of course a subspace of the 20-dimensional space considered above. Unfortunately, all basis vectors given above do not fulfill the additional orthogonality constraints, hence we have to find a completely new set of basis vectors. 6 of them are of the form $\beta_i = [b_1, -b_1, b_1]', \beta_{i+1} = [-b_1, 0, 0]', i = 0, \ldots, 5$. These vectors, with different orders of lagging, linearly combine the first difference of the first component of the output with the second and third component of y_t. The mechanism is similar to the mechanism in the example discussed in Section 2. In these relationships the linear combination of the second and third component of y_t is chosen so as to eliminate the third state component, which is not cointegrated with the second (and the first) state component. This relationship is the only essential polynomial cointegrating (linear in z) relationship for the unit root $z = 1$ that reduces the integration order from 2 to 0. The remaining 12 basis vectors are additional reformulations of the same fact, using different lags of the components of y_t and therefore of the components of $x_{t,0}$. 5 non-minimum degree relationships are given by $\beta_i = [b_1, -2b_1, 2b_1]', \beta_{i+2} = [-b_1, 0, 0]', i = 0, \ldots, 4$. Thus, still 7 relationships are missing: 6 of these are found from $\beta_i = [b_1, -b_1, 0]', \beta_{i+1} = [-b_1, 0, b_1]', i = 0, \ldots, 5$ which means that only one polynomial remains to be found. It is given by e.g. $\vec{\beta}' = [3, 5, 4, 2, 5, 4, 1, 5, 4, 0, 5, 4, -1, 5, 4, -2, 5, 4, -3, -58, 4]$. Taking the remainder of this vector polynomial divided by $(1-z)^2$ one obtains $[28-28z, 245-273z, -56+84z]'$. This latter vector is equal to $28[1-z, -1, 1]' + (1-z)[0, 273, -84]'$. Hence also this vector does not add any essential new cointegrating relation. Summing up, this implies that all minimum degree non-trivial polynomial cointegrating relationships that reduce the integration order at $z = 1$ from 2 to 0 and do not change the integration order at $z = i$ are constant scalar multiples of $[1-z, -1, 1]'$.

We do not repeat the same construction for the complex conjugate unit roots, but instead will use Theorem 3 and expression (14) to verify that for the example at hand there exist to all possible cointegration orders non-trivial polynomial cointegrating vectors. This is confirmed in Table 1, which presents the values of (14) for all configurations of cointegration structures. All values are nonzero. To give an example about the content of the left panel of the table, consider the cointegration order $((0, 2, 2), (\pi/2, 2, 1))$: Since static, and therefore non-trivial, cointegrating relationships that reduce the integration order at $\pm i$ from 2 to 1 and that do not reduce the integration order at $z = 1$, e.g. $[b_1, 0, b_3]', b_1 \neq 0$

	Non-minimum degree				minimum pol. degree 2		
Int. order	$z = 1$	2	1	0	2	1	0
$z = \pm i$							
2	$z = 1$	x	61.42	27.84	x	0	0
	$z = i$	x	74.97	50.15	x	0	0
1	$z = 1$	77.95	147.92	60.29	0	2.45	2.87
	$z = i$	64.16	148.95	91.96	0	0	2.87
0	$z = 1$	48.52	88.51	33.77	0	3.69	1.21
	$z = i$	36.67	82.42	46.06	0	2.61	2.76

Table 1: Expression (14) for all different cointegration orders: Columns correspond to unit root $z = 1$ and rows to $z = \pm i$. First row: expression for $z = 1$, second row: expression for $z = i$. Left panel corresponds to the existence of non-trivial polynomial cointegrating vectors of order 6 (not necessarily minimum degree). The right panel corresponds to the existence of minimum degree polynomials of degree two, to give just one example.

exist, the existence of polynomial cointegration of order $((0, 2, 2), (\pi/2, 2, 1))$ is trivially verified. The value of (14) is equal to approximately 64.16 at $z = i$. The right hand side of the table considers minimum degree polynomials of polynomial degree two as an example. E.g. the right panel shows, that there are no minimum degree polynomial cointegrating vectors of polynomial degree two, which leave the integration order at $z = 1$ unchanged and reduce the integration at $\pm i$. This since all polynomials of degree two that achieve this are non-minimum degree or trivial. However, there are non-trivial minimum degree polynomials for all cases, where the order of integration at both roots is reduced.

As a final remark we note that the set of all polynomial cointegrating relationships that eliminate all nonstationarities, in the sense that the resulting $\beta(z)'y_t$ is – up to a linearily deterministic process – stationary, is given by the orthogonal complement of the space spanned by the columns of $\tilde{\mathcal{O}}_{\mathbb{R},6}$, i.e. it can be described as a 14-dimensional space. In this space again trivial and non-minimum degree polynomials are contained. The existence of non-trivial cointegrating relationships that transform y_t to stationarity can as above be verified by computing the value of condition (14), resulting in 33.77 at $z = 1$ e.g (cf. Table 1). Moreover, also minimum degree non-trivial polynomials of degree two exist, which reduce the process to stationarity (cf. the value 1.21 at $z = 1$ in the corresponding entry in the right panel of Table 1).

9 Summary and Conclusions

In this paper we have demonstrated that the state space framework is very convenient for representing higher order integrated systems and for discussing cointegration and polynomial cointegration. The analysis is based on the canonical state space representation for processes with a rational transfer function developed in Bauer and Wagner (2003). Compared to available representation results for polynomial cointegration in the equivalent ARMA framework, the discussed state space representation offers a couple of advantages. Firstly, all polynomial cointegrating relationships can be found from simple orthogonality constraints. Secondly, the developed system representation is very instructive with respect to the integration and cointegration dynamics in the system. The state (cf. Theorem 1) is partitioned into blocks of components that are only integrated at one unit root, respectively, in a real representation, at one pair of complex conjugate unit roots. Within these blocks further structure concerning the integration properties between the state components integrated of different orders is directly visible within the coordinates corresponding to Jordan chains. This latter property has been shown to be at the heart of polynomial cointegration. Thus, this fact that the state space representation makes these *relationships along the chains* clearly visible is the main reason for the representation's elegance and simplicity in displaying polynomial cointegration. The set of orthogonality constraints can not only be used to easily find all polynomial cointegrating relationships of a certain order, but also to find all polynomial cointegrating relationships of a certain maximum polynomial degree, e.g. all linear cointegrating relationships. We have also discussed how to find all minimum degree non-trivial polynomial cointegrating relationships, which are the only ones considered to be *relevant* for the reasons discussed. Simple additional orthogonality constraints lead to the required classification.

The representation furthermore directly implies that the output is decomposed in a Granger type representation, i.e. decomposed as the sum of stochastic trends respectively cycles integrated of different orders plus the stationary components. Note that as a further property of the representation the (polynomial) cointegrating spaces only depend upon the defined state space unit root structure and the matrix C. This type of result compares, we think, favorably with the relatively complicated representation result derived in Gregoir (1999) based on an ARMA representation of the underlying stochastic process.

The disaggregated interpretation of the state and system matrices becomes even more intuitive, when one focuses on only one of the system blocks corresponding to only one unit root at a time, see the discussion in the appendix: Reparameterizing the polynomial cointegrating relationship in powers of the differencing filter corresponding to the unit root under study and a finer disaggregation of the state components and system matrices, highlights the interplay between differencing and orthogonality constraints that leads to polynomial cointegration. Since in our canonical representation the blocks corresponding to the different unit roots are separated, inspecting only one block at a time may be seen as a simple and intuitive alternative to investigating all blocks jointly, as is done in the theorems in the main text. When doing so, the issue of triviality in the sense of Definition 3 has to be investigated separately. It has to be verified that the polynomial cointegrating relationship is non-zero for at least one unit root z_k at which it leads to a reduction of the integration order. This has also been illustrated in the example in section 8.

Note finally that the representation results of this paper form an important basis for subsequent statistical analysis of polynomially cointegrated processes.

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Appendix: One Unit Root in Detail

In this appendix we discuss polynomial cointegration in detail when focusing only on one unit root, z_k say. Hence, we investigate only one block $\sum_{i=1}^{h_k} C_k^i x_{t,k}^i$. This allows to gain insights by reparameterizing the polynomial $\beta(z) = \sum_{j=0}^{q} \beta_j z^j$ as $\sum_{j=0}^{q} \tilde{\beta}_j (1 - \bar{z}_k z)^j$, where the latter representation of $\beta(z)$ is referred to as $\tilde{\beta}(z)$ below. In this appendix, mainly for notational simplicity, we focus on complex cointegration, therefore the coefficients $\beta_j \in \mathbb{C}^s$ are assumed to be complex rather than real. The arguments for real valued cointegrating relationships are analogous. We follow here at a general level the strategy outlined in the example in section 5.

A key observation made in the main text is the fact that in the developed canonical representation the components of the state (for the given unit root block) are linked in chains of increasing (complex) integration orders and via simple polynomial cointegrating relationships. The latter occur corresponding to the components of $x_{t,k}$ that are linked in a Jordan chain. Note that in the canonical representation the state components are ordered according to the complex integration order and not along Jordan chains, this has to be taken into account in the discussion and the notation.
The fact that the state components belonging to the same Jordan chain are linked via increasing integration orders is the key to understand polynomial cointegration. Thus, the integration properties of the state have to be discussed in full detail first. This requires a finer structural investigation of the blocks and their components. To this end partition $x^i_{t,k} = [(x^{i,i}_{t,k})', (x^{i,i-1}_{t,k})', \ldots, (x^{i,1}_{t,k})']'$ where $x^{i,m}_{t,k} \in \mathbb{C}^{d^k_m - d^k_{m-1}}$ for $m = 1, \ldots, i$ and $i = 1, \ldots, h_k$. Accordingly introduce also $C^{i,m}_k \in \mathbb{C}^{s\times(d^k_m - d^k_{m-1})}$, $B^{i,m}_k \in \mathbb{C}^{(d^k_m - d^k_{m-1})\times s}$ and for notational reasons $d^k_{h_k+1} = 0$ and $x^{h_k+1,m}_{t,k} = 0$. Then, it follows from equation (6) that $x^{i,m}_{t+1,k} - z_k x^{i,m}_{t,k} = x^{i+1,m}_{t,k} + B^{i,m}_k \varepsilon_t$ for $m = 1, \ldots, i$ and $i = 1, \ldots, h_k$. To summarize notation: The first superscript i refers to the integration order of $x^{i,m}_{t,k}$, given by $h_k - i + 1$, and the second superscript denotes the position of the state component in a Jordan chain, e.g. $m = 1$ indicates that this component corresponds to an eigenvector. Note that this is a refinement of the disaggregation undertaken in Theorem 1, where the state components are only disaggregated in components corresponding to eigenvectors (referred to with E) and to generalized eigenvectors (referred to as G). Thus, all components with second superscript $m > 1$ are contained in the G-components of the more crude disaggregation required for the results in Theorem 1.
With this disaggregated representation of the block, the effect of pre-multiplying y_t with a vector polynomial $\beta(z)$ on the integration order of the product $\beta(z)'y_t$ at the unit root z_k can be analyzed in detail. Note that the maximum polynomial degree required for complex unit roots, since we are using a complex valued representation, is given by $h_k - 1$. Hence, it suffices to set

$q = h_k - 1$:[2]

$$
\begin{aligned}
\sum_{r=0}^{h_k-1} \tilde{\beta}_r'(1-z_k z)^r \sum_{i=1}^{h_k} C_k^i x_{t,k}^i &= \sum_{i=1}^{h_k} \sum_{r=0}^{h_k-1} \sum_{m=1}^{i} \tilde{\beta}_r' C_k^{i,m} (1-z_k z)^r x_{t,k}^{i,m} \\
&= \sum_{i=1}^{h_k} \sum_{r=0}^{h_k-i} \sum_{m=1}^{i} \tilde{\beta}_r' C_k^{i,m} x_{t,k}^{i+r,m} + \nu_t \\
&= \sum_{i=1}^{h_k} \sum_{l=i}^{h_k} \sum_{m=1}^{i} \tilde{\beta}_{l-i}' C_k^{i,m} x_{t,k}^{l,m} + \nu_t \\
&= \sum_{l=1}^{h_k} \sum_{i=1}^{l} \tilde{\beta}_{l-i}' \left(\sum_{m=1}^{i} C_k^{i,m} x_{t,k}^{l,m} \right) + \nu_t \\
&= \sum_{l=1}^{h_k} \underbrace{\left(\sum_{m=1}^{l} \left(\sum_{i=m}^{l} \tilde{\beta}_{l-i}' C_k^{i,m} \right) x_{t,k}^{l,m} \right)}_{\text{Integration order} \leq h_k - l + 1} + \nu_t \qquad (15) \\
&= \sum_{l=1}^{h_k} I(\tilde{\beta}(z)' y_t, z_k, l) + \nu_t
\end{aligned}
$$

where $l = i + r$ and ν_t collects all asymptotically stationary parts. Representation (15) allows to directly infer the polynomial cointegrating relationships, as it collects the terms corresponding to the different integration orders. According to Property 3 of Theorem 1, the term $\sum_{m=1}^{l} \left(\sum_{i=m}^{l} \tilde{\beta}_{l-i}' C_k^{i,m} \right) x_{t,k}^{l,m}$ has integration structure $((\omega_k, h_k - l + 1))$, unless $\sum_{i=m}^{l} \tilde{\beta}_{l-i}' C_k^{i,m} = 0$ for $m = 1, \ldots, l$. From (15) now directly the polynomial cointegrating relationships can be read off. The complex integration order of y_t at the unit root z_k is reduced from h_k to $h_k - l_0$ say, if and only if due to pre-multiplication with $\beta(z)$ the terms $I(\tilde{\beta}(z)' y_t, z_k, 1), \ldots, I(\tilde{\beta}(z)' y_t, z_k, l_0)$ are all equal to 0, but $I(\tilde{\beta}(z)' y_t, z_k, l_0 + 1) \neq 0$. Investigating the individual terms, i.e. the corresponding sums of products that are required to equal zero, shows that these conditions can be conveniently rewritten as orthogonality constraints using the matrix:

$$
\begin{aligned}
M_k &= \left[\begin{array}{c|cc|ccc|c|cccc}
C_k^{1,1} & C_k^{2,1} & C_k^{2,2} & C_k^{3,1} & C_k^{3,2} & C_k^{3,3} & \cdots & C_k^{h_k,1} & C_k^{h_k,2} & \cdots & C_k^{h_k,h_k} \\
0 & C_k^{1,1} & 0 & C_k^{2,1} & C_k^{2,2} & 0 & & C_k^{h_k-1,1} & C_k^{h_k-1,2} & & 0 \\
\vdots & 0 & 0 & C_k^{1,1} & 0 & \vdots & & \vdots & \vdots & & \vdots \\
0 & \vdots & \vdots & 0 & \vdots & 0 & & C_k^{1,1} & 0 & \cdots & 0
\end{array} \right], \\
&= \left[\begin{array}{cccc} M_k^1 & M_k^2 & \ldots & M_k^{h_k} \end{array} \right].
\end{aligned}
$$

The above matrix is the general case equivalent of the matrix in (11) in Section 5. As in that example orthogonality to certain blocks determines polynomial cointegration. M_k is composed

[2] In real valued representations and for real valued cointegrating polynomials the maximum required polynomial degree is equal to $2h_k - 1$ when focusing on only one particular pair of complex conjugate unit roots.

of h_k block-rows (corresponding to the polynomial degrees from 0 to $h_k - 1$, higher polynomial degrees just lead to null block-rows at the bottom of the matrix). Orthogonality to the block-columns, denoted by $M_k^1, \dots, M_k^{h_k}$, determines the polynomial cointegration orders, using the vector notation $\tilde{\beta} = [\tilde{\beta}_0', \tilde{\beta}_1', \dots, \tilde{\beta}_{h_k-1}']'$ for $\tilde{\beta}(z)$. If e.g. $\tilde{\beta}$ is only orthogonal to M_k^1, but not to M_k^2, then the complex integration order of $\beta(z)'y_t$ at the unit root z_k drops from h_k to h_k-1. This condition just amounts to the requirement $\tilde{\beta}_0' C_k^{1,1} = 0$. Obviously there exist many solutions in terms of polynomials up to degree h_k-1. However, since we developed the polynomials in powers of $(1 - \bar{z}_k z)$ all polynomials with $\tilde{\beta}_0 = 0$ are seen to be trivial solutions. Also, all polynomials with $\tilde{\beta}_0 \neq 0$, $\tilde{\beta}_0' C_k^{1,1} = 0$ and $\tilde{\beta}_j \neq 0$ for some $j \geq 1$ are directly seen to be non-minimum degree. The same type of argument can now be repeated for relationships that reduce the integration order by 2, via orthogonality to the first two block-columns. All polynomials with $\tilde{\beta}_0 = \tilde{\beta}_1 = 0$ are seen to be trivial solutions, as are all polynomials of the form $\tilde{\beta}_0 = 0$ and $\tilde{\beta}_1' C_k^{1,1} = 0$. Hence, the relevant cointegrating relationships are given by static relationships with $\tilde{\beta}_0' \begin{bmatrix} C_k^{1,1} & C_k^{2,1} & C_k^{2,2} \end{bmatrix} = 0$ and by linear relationships of the form $\tilde{\beta}_0' \begin{bmatrix} C_k^{1,1} & C_k^{2,2} \end{bmatrix} = 0$, $\tilde{\beta}_0' C_k^{2,1} + \tilde{\beta}_1' C_k^{1,1} = 0$ and $\tilde{\beta}_1' C_k^{1,1} \neq 0$. If the latter additional condition $\tilde{\beta}_1' C_k^{1,1} \neq 0$ were not satisfied, the relationship would not be of minimum degree. Thus, orthogonality to the corresponding non-zero blocks in M_k and zero coefficients in $\tilde{\beta}$ corresponding to the 0 entries in the corresponding block-row(s) of M_k leads directly to the relevant polynomial cointegrating relationships for any given unit root, respectively for any given pair of complex conjugate unit roots.

Note finally once more that non-triviality of a polynomial cointegrating relationships requires it to be non-trivial only at *one* of the unit roots where it leads to a reduction of the integration order. Thus, for uncovering all non-trivial polynomial cointegrating relationships the behavior of the polynomial cointegrating relationship has to be investigated at all unit roots where it leads to a reduction of the integration order.